

Laplace Operators on Fractals and Related Functional Equations

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Abstract. We give an overview over the application of functional equations, namely the classical Poincaré and renewal equations, to the study of the spectrum of Laplace operators on self-similar fractals. We compare the techniques used to those used in the euclidean situation. Furthermore, we use the obtained information on the spectral zeta function to define the Casimir energy of fractals. We give numerical values for this energy for the Sierpiński gasket.

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1. Introduction

1.1. Historical perspective

Strange objects, which are poorly characterised by their topological dimension, such as the Weierstrass continuous, but nowhere differentiable function, the van Koch curve, the Sierpiński gasket (see Figure 1), the Sierpiński carpet (see Figure 4), etc., have been familiar in mathematics for a long time. Now such objects are called *fractals*.

The word “fractal” was coined by Mandelbrot in the 1970s. His foundational treatise [1] contains a great diversity of examples from mathematics and natural sciences. Coastlines, topographical surfaces, turbulence in fluid, and so on are just few instances from the variety of natural objects that may be described as fractals. There is no generally agreed exact definition of the word “fractal”.

The fractals studied in the context of analysis on fractals are all *self-similar*. Moreover, we deal here mainly with *finitely ramified* fractals, i.e., fractals that can be disconnected by removing a specific finite number of points.

Initial interest in processes (analysis) on fractals came from physicists working in the theory of *disordered media*. It turns out that heat and wave transfer in disordered media (such as polymers, fractured and porous rocks, amorphous semiconductors, etc.) can be adequately modelled by means of fractals and random walks on them. (See the initial papers by Alexander and Orbach [2], Rammal and Toulouse [3]. See also the survey by Havlin and Ben-Avraham [4] and the book by the same authors [5] – for an overview of the now very substantial physics literature and bibliography.)

Motivated by these works, mathematicians got interested in developing the of “analysis on fractals”. For instance, in order to analyse how heat diffuses in a material with fractal structure, one needs to define a “heat equation” and a “Laplacian” on a fractal. The problem contains somewhat contradictory factors. Indeed, fractals like the Sierpiński gasket, or the van Koch curve do not have any smooth structures and one cannot define differential operators on them directly. We will comment on the difference between the euclidean and the fractal situation throughout this paper.

1.2. Probabilistic approach

In the mid 1980s probabilists solved the problem by constructing “Brownian motion” on the Sierpiński gasket. Goldstein [6], Kusuoka [7], and a bit later Barlow and Perkins [8], independently took the first step in the mathematical development of the problem. Their method of construction is now called the *probabilistic approach*. Namely, they considered a sequence of random walks $X^{(n)}$ on graphs G_n , which approximate the Sierpiński gasket Γ , and showed that by taking a certain scaling factor, those random walks converge to a diffusion process on the Sierpiński gasket.

In order to obtain a nontrivial limit, the time scale for each step should be the average time for the random walk on G_{n+1} , starting from a point in G_n , to arrive at a point in G_n , except for the starting point. By the self-similarity and symmetry of Γ this

average time τ is independent of n and it is equal to the average time for the random walk on G_1 , starting from a to arrive at either b or c . (Here a, b, c , are the vertices of the initial equilateral triangle G_0 .) Elementary calculations show that $\tau = 5$. According to [6, 7, 8] the processes $2^{-n}X^{(n)}([5^n t])$ weakly converge (as $n \rightarrow \infty$) to a non-trivial limit X_t on Γ , which is called Brownian motion on Γ . In this approach the Laplacian is defined as an *infinitesimal generator* of X_t .

An important observation was made by Barlow and Perkins in [8]. Let $Z_n = T_1^{n,0}$ be the *first hit* time by $X^{(n)}$ on G_0 . Then Z_n is a *simple branching process*. Its off-spring distribution η has the generating function $q(z) = \mathbb{E}(z^\eta) = z^2/(4-3z)$ and, in particular, $\mathbb{E}(\eta) = q'(1) = 5$. Thus, Z_n is a *super-critical* branching process. It is known (see [9]) that in this case $5^{-n}Z_n$ tends to a limiting random variable Z_∞ .

The moment generating function of this random variable

$$f(z) = \mathbb{E}e^{-zZ_\infty}$$

satisfies the functional equation

$$f(\lambda z) = q(f(z)), \tag{1}$$

which is the *Poincaré equation* (see also Section 5 below).

The Poincaré equation associated with the Brownian motion turns out to be a very useful tool in the study of the detailed properties (for example, heat kernel) of the Brownian motion.

Lindstrøm [10] extended the construction of the Brownian motion from the Sierpiński gasket to more general *nested fractals*, (which are finitely ramified self-similar fractals with strong symmetry). The Lindstrøm *snowflake* is a typical example of a nested fractal (see Figure 2).

Readers may refer to Barlow's lecture notes [11] for a self-contained survey of the probabilistic approach.

1.3. Anomalous diffusion

It has been discovered in an early stage already (see [2, 3, 6, 7, 8]) that diffusion on fractals is *anomalous*, different than that in a regular space. For a regular diffusion, or (equivalently) a simple random walk in all integer dimensions d , mean-square displacement is proportional to the number of steps n : $\mathbb{E}(X_n)^2 = cn$ (Fick's law, 1855). On the other hand, in the case of the Sierpiński gasket $\mathbb{E}^x(X_n - x)^2 \asymp n^{2/\beta}$, where \asymp means that the ratio between the two sides is 'bounded above and below by positive constants' and $\beta = \lg 5 / \lg 2$ is called the *walk dimension*.

This slowing down of the diffusion is caused, roughly speaking, by the removal of large parts of the space.

De Gennes [12] was amongst the first, who realised the broad importance of anomalous diffusion and coined the suggestive term "the ant in the labyrinth", describing the meandering of a random walker in percolation clusters.

1.4. Analytic approach

The second approach, based on difference operators, is due to Kigami [13]. Instead of the sequence of random walks, one can consider a sequence of discrete Laplacians on a sequence of graphs, approximating the fractal. It is possible to prove that under a proper scaling these discrete Laplacians would converge to an “well-behaved” operator with dense domain, called the Laplacian on the Sierpiński gasket. This alternative approach is usually called the *analytic approach*.

Later it was extended by Kigami [14, 15, 16] to more general class of fractals – *post critically finite self-similar sets* (p.c.f), which roughly correspond to finitely ramified self-similar fractals.

The two approaches described above are complementary to each other.

The advantage of the analytic approach is that one gets concrete and direct description of harmonic functions, Laplacians, Dirichlet forms, etc. (See also [17, 18].)

On the other hand, however, the probabilistic approach is better suited for the study of heat kernels. Moreover, this approach can be applied to *infinitely ramified* self-similar fractals, which include the Sierpiński carpet, as a typical example (cf., [19]).

1.5. Functional equations in the analysis on fractals

One important example, where the Poincaré equation arises in connection with analysis on fractals, has been mentioned already at the end of Section 1.2.

Further applications of functional equations in this field are related to spectral zeta function ζ_Δ of the Sierpiński gasket and other more general fractals having *spectral decimation*. The phenomenon of spectral decimation was first observed and studied by Fukushima and Shima ([17, 20, 21]) and further progress has been made by Malozemov and Teplyaev [22] and Strichartz [23].

The definition of spectral decimation is given in Section 4.2 (see Definition 8 below). It implies, in particular, that eigenvalues of the Laplacian Δ on the fractal, which admits spectral decimation, can be calculated by means of a certain polynomial $p(z)$, or rational function $R(z)$. Hence, the spectral zeta function ζ_Δ may be defined by means of iterations $p^{(n)}$ of p , or $R^{(n)}$ of R (see [24]).

The above mentioned iteration process, as is well known in iteration theory, may be conveniently described by the corresponding Poincaré equation:

$$\Phi(\lambda z) = p(\Phi(z)), \quad (2)$$

where $\lambda = p'(0) > 1$ (see, for example, Beardon [25] or Milnor [26]).

Using that, in Section 4.4, we obtain the meromorphic continuation of the zeta function ζ_Δ into the whole complex plane on the basis of the knowledge of the asymptotic behaviour of the Poincaré function Φ in certain angular regions.

The poles of the spectral zeta function are called the *spectral dimensions* (see [27, 28, 29]). For the physical consequences of complex dimensions of fractals – see

([30, 31]). In Section 4.5.1, we use the Poincaré function Φ for the calculation of *Casimir energy* on a fractal.

Finally, one can expect that functional equations with rescaling naturally come about from problems, where renormalisation type arguments are used to study self-similarity. Furthermore, functional equations, in contrast with differential ones, do not require any *smoothness* of solutions; they may possess nowhere differential solutions, for example.

1.6. Notes and remarks

There is now a number of excellent books, lecture notes and surveys on different aspects of analysis and probability on fractals [5, 11, 16, 29, 32, 33, 34, 35, 36].

It is next to impossible to describe all activities in this area. The objective of the present paper is different. We restrict ourselves to a brief overview of various approaches in the study of the Laplacian and its spectral properties on certain self-similar fractals, and we put much emphasis on the deep connection between the latter problem and functional equations with rescaling, and the classical Poincaré equation, in particular.

The Poincaré equation plays a very important role in the mathematical theory of dynamical systems and in iteration theory, in particular, but is still much less known in the physics literature. Hopefully, the current presentation of this realm of problems intended for a general audience may fill this gap.

We begin our presentation of the Poincaré equation in Section 5 with a brief description of the general case, when the coefficients of the polynomial P and the scaling factor λ are *complex* numbers, and only afterwards turn to a detailed discussion of the *real* case. So far, the real case only has arisen in analysis on fractals. However, we think that the general case is interesting on its own, and, probably, will also find applications in the future.

In this overview we do not or only cursorily touch the following important topics:

- analysis on infinitely ramified fractals such as the Sierpiński carpet [19, 37]. The very recent progress that has been made in proving the uniqueness of the diffusion on the Sierpiński carpets [38] provided a unification of the different approaches to diffusion on this class of fractals.
- heat kernel long time behaviour and Harnack inequalities on general underlying spaces, including Riemannian manifolds, graphs and fractals, as special cases [39, 40, 41, 42]
- analysis by means of potential theory and functional spaces (Sobolev or Besov spaces) techniques [33, 43, 44, 45, 46, 47, 48]

We refer the reader to the original papers.

2. Fractals and iterated function systems

Let us first recall that for any finite set of linear contractions on \mathbb{R}^d

$$F_i(\mathbf{x}) = \mathbf{b}_i + \mathbf{A}_i(\mathbf{x} - \mathbf{b}_i), \quad i = 1, \dots, m$$

with fixed points \mathbf{b}_i and contraction matrices \mathbf{A}_i ($\|\mathbf{A}_i\| < 1$, $i = 1, \dots, m$) there exists a unique compact set $K \subset \mathbb{R}^d$ satisfying

$$K = \bigcup_{i=1}^m F_i(K). \quad (3)$$

In general, the set K obtained as fixed point in (3) is a “fractal” set (cf. [49, 50]). If the matrices \mathbf{A}_i are only assumed to be contracting, finding the Hausdorff-dimension of K is a rather delicate (cf. [51, 52]) problem.

For this paper we will make the following additional assumptions on the family of contractions $F = \{F_i \mid i = 1, \dots, m\}$:

- (i) the matrices \mathbf{A}_i are similitudes with factors $\alpha_i < 1$:

$$\forall \mathbf{x} \in \mathbb{R}^d : \|\mathbf{A}_i \mathbf{x}\| = \alpha_i \|\mathbf{x}\|$$

- (ii) F satisfies the open-set-condition (cf. [49, 53]), namely, there exists a bounded open set \mathcal{O} such that

$$\bigcup_{i=1}^m F_i(\mathcal{O}) \subset \mathcal{O}$$

with the union disjoint.

Assuming that the maps in F are similitudes and satisfy the open-set-condition, the Hausdorff-dimension of the compact set K given by (3) equals the unique positive solution $s = \rho$ of the equation

$$\sum_{i=1}^m \alpha_i^s = 1. \quad (4)$$

The projection map

$$\begin{aligned} \pi : \{1, \dots, m\}^{\mathbb{N}} &\rightarrow K \\ (\varepsilon_1, \varepsilon_2, \dots) &\mapsto \lim_{n \rightarrow \infty} F_{\varepsilon_1} \circ F_{\varepsilon_2} \circ \dots \circ F_{\varepsilon_n}(\mathbf{x}) \end{aligned} \quad (5)$$

(the limit is independent of $\mathbf{x} \in \mathbb{R}^d$) defines a “parametrisation” of K . If $\{1, \dots, m\}^{\mathbb{N}}$ is endowed with the infinite product measure μ given by

$$\mu(\{(\varepsilon_1, \varepsilon_2, \dots) \mid \varepsilon_1 = i_1, \varepsilon_2 = i_2, \dots, \varepsilon_k = i_k\}) = (\alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_k})^\rho, \quad (6)$$

then π is a μ -almost sure bijection. The normalised ρ -dimensional Hausdorff-measure on K is then given by

$$\mathcal{H}_K^\rho(E) := \frac{\mathcal{H}^\rho(E \cap K)}{\mathcal{H}^\rho(K)} = \pi_*(\mu)(E) = \mu(\pi^{(-1)}(E)). \quad (7)$$

This is the unique normalised measure satisfying (cf. [53, Theorem 28])

$$\mathcal{H}_K^\rho(E) = \sum_{i=1}^m \alpha_i^\rho \mathcal{H}_K^\rho(F_i^{(-1)}(E)). \quad (8)$$

By the above discussion the functions

$$\varepsilon_n(\mathbf{x}) = (\pi^{(-1)}(\mathbf{x}))_n$$

are defined \mathcal{H}_K^ρ -almost everywhere on K . We set

$$\varepsilon_n(\mathbf{x}) = \min(\pi^{(-1)}(\{\mathbf{x}\}))_n \quad (9)$$

to define them everywhere. The set K together with the address space $\Sigma = \{1, \dots, m\}^\mathbb{N}$ and the maps F_i ($i = 1, \dots, m$) defines a *self-similar structure* $(K, \Sigma, (F_i)_{i=1}^m)$.

In order to allow sensible analysis on the fractal set K , we need some further properties. Especially, since we will later study diffusion processes, we need K to be connected. On the other hand, the techniques introduced later require a finite ramification property usually called post-critical finiteness (p. c. f.).

Connectivity of K is characterised by the fact that for any pair (i, j) , there exist $i_1, i_2, \dots, i_n \in \{1, \dots, m\}$ with $i = i_1$ and $j = i_n$ such that $F_{i_\ell}(K) \cap F_{i_{\ell+1}}(K) \neq \emptyset$ for $\ell = 1, \dots, n-1$ (cf. [54]).

Definition 1 Let $(K, \Sigma, (F_i)_{i=1}^m)$ be a self-similar structure. Then the set

$$C = \pi^{-1} \left(\bigcup_{i \neq j} F_i(K) \cap F_j(K) \right)$$

is called the *critical set* of K . The *post-critical set* of K is defined by

$$P = \bigcup_{n=1}^{\infty} \sigma^n(C),$$

where $\sigma : \Sigma \rightarrow \Sigma$ denotes the shift map on the address space Σ . If P is a finite set, then $(K, \Sigma, (F_i)_{i=1}^m)$ is called *post-critically finite* (p. c. f.). This is equivalent to the finiteness of C together with the fact that all points of C are ultimately periodic.

The following sequence V_m of finite sets will be used in Section 3.5 to define a sequence of electrical networks giving a harmonic structure on K . For more details we refer to [16, Chapter 1]

Definition 2 Let $(K, \Sigma, (F_i)_{i=1}^m)$ be a post-critically finite self-similar structure and P its post-critical set. Let $V_0 = \pi(P)$ and define V_m iteratively by

$$V_{n+1} = \bigcup_{i=1}^m F_i(V_n).$$

The sets V_n are then finite, increasing ($V_n \subset V_{n+1}$) and

$$K = \overline{\bigcup_{n \geq 0} V_n}.$$

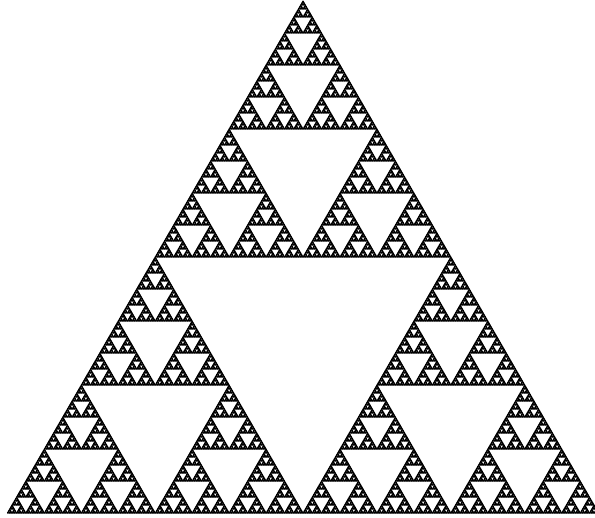


Figure 1. The Sierpiński gasket; the points of V_0 are the vertices of the triangle

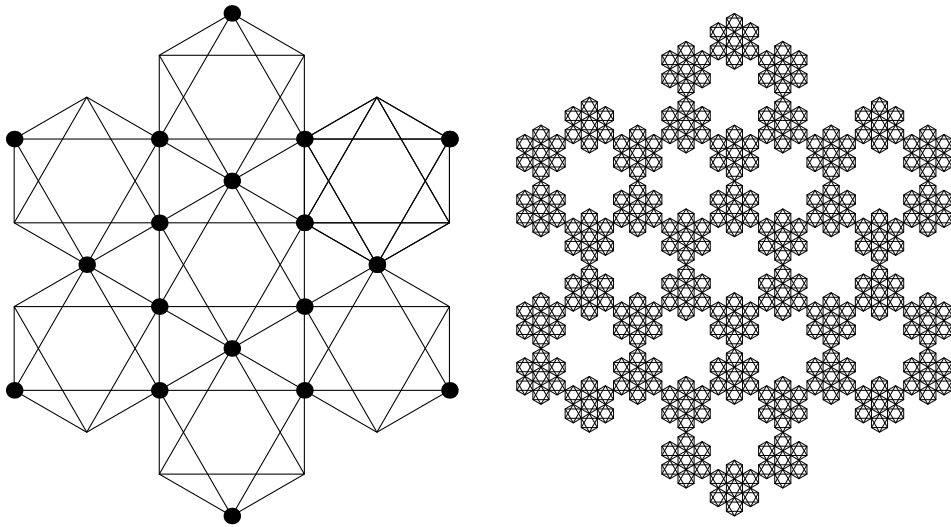


Figure 2. The Lindström snowflake with the corresponding set V_0

3. Laplace operators on fractals

3.1. Laplace operators on compact manifolds

Before we introduce the Laplace operator on certain classes of self-similar fractals, let us shortly discuss the situation in the manifold case, because this gives the motivation for the different approaches in the case of fractals. Let M be a compact Riemannian manifold with a Riemannian metric g given as a quadratic form g_x on the tangent space $T_x M$ for $x \in M$. As usual, we assume that the dependence of g_x on x is differentiable. Then the quadratic form g_x defines an isomorphism α_g between the tangent space $T_x M$ and its dual $T_x^* M$ (and thus on the tangent bundle TM and the cotangent bundle $T^* M$) by $\alpha_g(v)w = g_x(v, w)$ for $v, w \in T_x M$. This defines the gradient of a function f as $\text{grad } f = \alpha_g^{-1}(df)$. Define the divergence of a vector field X as the negative formal

adjoint of grad with respect to the scalar product $\langle X, Y \rangle_{L^2(M)} = \int_M g_x(X, Y) \, d \, \text{vol}(x)$:

$$\langle X, \text{grad } f \rangle_{L^2(M)} = -\langle \text{div } X, f \rangle_{L^2(M)}.$$

The Laplace operator is then defined as (cf. [55, 56])

$$\Delta f = \text{div grad } f. \quad (10)$$

By definition this operator is self-adjoint and thus has only non-positive real eigenvalues by $\langle \Delta f, f \rangle_{L^2} = -\langle \text{grad } f, \text{grad } f \rangle_{L^2} \leq 0$.

Based on the above approach, a corresponding energy form (“Dirichlet form”, cf. [57]) can be defined

$$\mathcal{E}(u, v) = \int_M g_x(\text{grad } u, \text{grad } v) \, d \, \text{vol}(x) = \int_M g_x(du, dv) \, d \, \text{vol}(x),$$

which lends itself to a further way of defining a Laplace operator via the relation

$$\mathcal{E}(u, v) = -\langle \Delta u, v \rangle_{L^2(M)}. \quad (11)$$

Geometrically, the Laplace operator measures the deviation of the function f from the mean value. More precisely, let $S(x, r) = \{y \in M \mid d(x, y) = r\}$ denote the ball of radius r in M (in the Riemannian metric). Then

$$\Delta f = 2n \lim_{r \rightarrow 0+} \frac{1}{r^2 \sigma(S(x, r))} \int_{S(x, r)} (f(y) - f(x)) \, d\sigma(y), \quad (12)$$

where n denotes the dimension of the manifold M and σ is the surface measure on $S(x, r)$. This also motivates the definition of the Laplace operator as limit of finite difference operators

$$\Delta f(x) = \lim_{r \rightarrow 0} \frac{1}{r^2} \left(\sum_{p \in N_r(x)} w_p f(p) - f(x) \right),$$

where $N_r(x)$ is a finite set of points at distance r from x and w_p are suitably chosen weights. Such approximations to the Laplace operator are the basis of the method of finite differences in numerical mathematics.

The Laplace operator can then be used to define a diffusion on M via the heat equation $\Delta u = \partial_t u$. The solution $u(t, x)$ of the initial value problem $u(0, x) = f(x)$ defines a semi-group of operators P_t by

$$u(t, x) = P_t f(x).$$

The semi-group property $P_{s+t} = P_s P_t$ comes from the uniqueness of the solution u and translation invariance with respect to t of the heat equation. From the heat semi-group P_t the Laplace operator can be recovered as the infinitesimal generator

$$\Delta f = \lim_{t \rightarrow 0+} \frac{P_t f - f}{t}, \quad (13)$$

which exists on a dense subspace of $L^2(M)$ under suitable continuity assumptions on the semi-group $(P_t)_{t \geq 0}$ (cf. [58]).

In the fractal situation none of the above approaches can be used directly to define a Laplace operator. The main reason for this is that there is no natural definition of derivative on a fractal. But the above approaches to the Laplace operator on a manifold can be used in the opposite direction:

- starting from a diffusion process that can be defined on fractals by approximating random walks. Then the Laplace operator can be defined as the infinitesimal generator. This is described in Sections 3.2 and 3.3.
- taking the limit of finite difference operators on graphs approximating the fractal gives a second possible approach to the Laplace operator, which is explained in Section 3.4.
- starting with a Dirichlet form \mathcal{E} gives a third possible approach, which is presented in Section 3.5.

3.2. Random walks on graphs and diffusion on fractals

The first idea to define a diffusion on a fractal was to define a sequence of random walks on approximating graphs and to synchronise time so that the limiting process is non-constant and continuous. This was the first approach to the diffusion process on the Sierpiński gasket given in [6, 7, 8] and later generalised to other “nested fractals” in [10]. Because of its importance for our exposition, we will explain it in some detail in this section. We will follow the lines of definition of self-similar graphs given in [59, 60] and adapt it for our purposes.

We consider a graph $G = (V(G), E(G))$ with vertices $V(G)$ and undirected edges $E(G)$ denoted by $\{x, y\}$. We assume throughout that G does not contain multiple edges nor loops. For $C \subset V(G)$ we call ∂C the vertex boundary, which is given by the set of vertices in $V(G) \setminus C$, which are adjacent to a vertex in C . For $F \subset V(G)$ we define the reduced graph G_F by $V(G_F) = F$ and $\{x, y\} \in E(G_F)$, if x and y are in the boundary of the same component of $V(G) \setminus F$.

Definition 3 *A connected infinite graph G is called self-similar with respect to $F \subset V(G)$ and $\varphi : V(G) \rightarrow V(G_F)$, if*

- (i) *no vertices in F are adjacent in G*
- (ii) *the intersection of the boundaries of two different components of $V(G) \setminus F$ does not contain more than one point*
- (iii) *φ is an isomorphism of G and G_F .*

A random walk on G is given by transition probabilities $p(x, y)$, which are positive, if and only if $\{x, y\} \in E(G)$. For a trajectory $(Y_n)_{n \in \mathbb{N}_0}$ of this random walk with $Y_0 = x \in F$ we define stopping times recursively by

$$T_{m+1} = \min \{k > T_m \mid Y_k \in F \setminus \{Y_{T_m}\}\}, \quad T_0 = 0.$$

Then $(Y_{T_m})_{m \in \mathbb{N}_0}$ is a random walk on G_F . Since the underlying graphs G and G_F are isomorphic, it is natural to require that $(\varphi^{-1}(Y_{T_m}))_{m \in \mathbb{N}_0}$ is the same stochastic process as $(Y_n)_{n \in \mathbb{N}_0}$. This requires the validity of equations for the basic transition probabilities

$$\mathbb{P}(Y_{T_{n+1}} = \varphi(y) \mid Y_{T_n} = \varphi(x)) = \mathbb{P}(Y_{n+1} = y \mid Y_n = x) = p(x, y). \quad (14)$$

These are usually non-linear rational equations for the transition probabilities $p(x, y)$. The existence of solutions of these equations has been the subject of several investigations, and we refer to [61, 62, 63, 64].

The process $(Y_n)_{n \in \mathbb{N}_0}$ on G and its “shadow” $(Y_{T_n})_{n \in \mathbb{N}_0}$ on G_F are equal, but they are on a different time scale. Every transition $Y_{T_n} \rightarrow Y_{T_{n+1}}$ on G_F comes from a path $Y_{T_n} \rightarrow Y_{T_{n+1}-1} \rightarrow \dots \rightarrow Y_{T_{n+1}}$ in a component of $V(G) \setminus F$. The time scaling factor between these processes is given by

$$\lambda = \mathbb{E}(T_{n+1} - T_n) = \mathbb{E}(T_1).$$

This factor is ≥ 2 by assumption (i) on F . More precisely, the relation between the transition time on G_F and the transition time on G is given by a super-critical ($\lambda > 1$) branching process, which replaces an edge $\{\varphi(x), \varphi(y)\} \in G_F$ by a path in G connecting the points x and y without visiting a point in $V(G) \setminus F$ (except for x and for y in the last step).

In order to obtain a process on a fractal in \mathbb{R}^d , we assume further that G is embedded in \mathbb{R}^d (i.e. $V(G) \subset \mathbb{R}^d$). The self-similarity of the graph is carried over to the embedding by assuming that there exists a $\beta > 1$ (the space scaling factor) such that $F = V(G_F) = \beta V(G)$. The fractal limiting structure is then given by

$$Y_G = \overline{\bigcup_{n=0}^{\infty} \beta^{-n} V(G)}.$$

Iterating this graph decimation we obtain a sequence of (isomorphic) graphs $G_n = (\beta^{-n} V(G), E(G))$ on different scales. The random walks $(Y_k^{(n)})_{k \in \mathbb{N}_0}$ on G_n are connected by time scales with the scaling factor λ . From the theory of branching processes (cf. [9]) it follows that the time on level n scaled by λ^{-n} tends to a random variable. From this it follows that $\beta^{-n} Y_{[t\lambda^n]}$ weakly tends to a (continuous time) stochastic process $(X_t)_{t \geq 0}$ on the fractal Y_G .

Under the assumption that the graph G has an automorphism group acting doubly transitively on the points of the boundary of every component of $G \setminus F$ (cf. [59, 60]), we consider the *generating function* of the transition probabilities $p_n(x, y) = \mathbb{P}(Y_n = y \mid Y_0 = x)$, the so called *Green function* of the random walk

$$G(z \mid x, y) = \sum_{n=0}^{\infty} p_n(x, y) z^n,$$

which can also be seen as the resolvent $(I - zP)^{-1}$ of the transition operator P . The replacement rules connecting the random walks on the graphs G and G_F result in a functional equation for the Green function

$$G(z \mid \varphi(x), \varphi(y)) = f(z) G(\psi(z) \mid x, y), \quad (15)$$

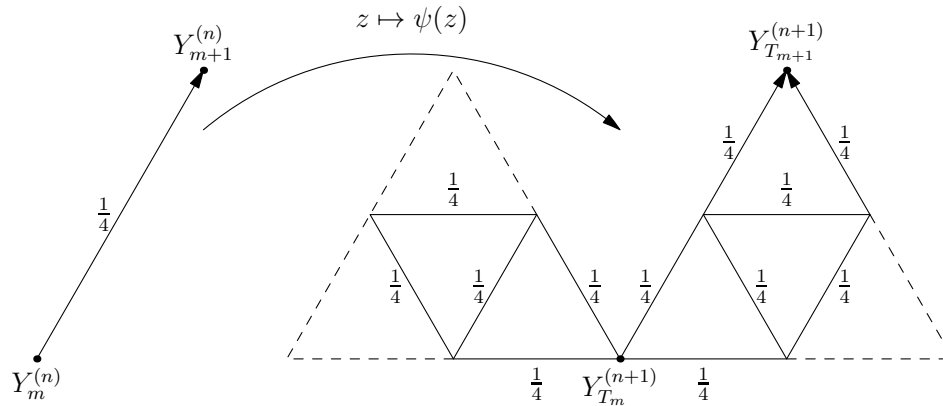


Figure 3. Transition between $Y_k^{(n)}$ and $Y_k^{(n+1)}$

where the rational function $\psi(z)$ is the probability generating function of the paths connecting two points $v_1, v_2 \in \varphi(V(G))$ without reaching any point in $\varphi(V(G)) \setminus \{v_1\}$. Then the time scaling factor is the expected number of steps needed for the paths counted by ψ , so we have

$$\lambda = \psi'(1).$$

The rational function $f(z)$ is the probability generating function of the paths starting and ending in $v_1 \in \varphi(V(G))$ without reaching any other point in $\varphi(V(G))$. Equation (15) becomes especially simple for a fixed point x of the map φ

$$G(z \mid x, x) = f(z)G(\psi(z) \mid x, x).$$

It was proved in [60] that under our conditions on the set F the map φ can have at most one fixed point.

The Koenigs function Φ of ψ around the fixed point $z = 1$ is given by

$$\Phi(\lambda z) = \psi(\Phi(z)), \quad \Phi(0) = 1, \quad \Phi'(0) = 1.$$

This can be used to linearise this functional equation and to obtain precise analytic information about $G(z \mid x, x)$ (cf. [60, 65]). From this the asymptotic behaviour of the transition probabilities $p_n(x, x)$ can be derived. In many examples these transition probabilities exhibit periodic fluctuations

$$p_n(x, x) \sim n^{-\frac{d_S}{2}} \left(\sigma(\log_\lambda n) + \mathcal{O}(n^{-1}) \right),$$

where σ is a continuous, periodic, non-constant function of period 1 (cf. [60]).

A first example of such graphs is the Sierpiński graph studied as an approximation to the fractal Sierpiński gasket. In this case we have for the probability generating function $\psi(z) = \frac{z^2}{4-3z}$ and $\lambda = 5$. The random walk on this graph was studied in [8] in order to define a diffusion on a fractal set. Self similarity of the graph and the fractal have been exploited further, to give a more precise description of the random walk [66] and the diffusion [67]. In [60] a precise description of the class of graphs in terms of their symmetries is given, which allow a similar construction. In [68] this analysis was

carried further to obtain results for the transition probabilities under less symmetry assumptions using multivariate generating functions.

In some examples (for instance the Sierpiński graph) it occurred that the function ψ was conjugate to a polynomial p , i.e.

$$\psi(z) = \frac{1}{p(\frac{1}{z})},$$

which allowed a study of the properties of the random walk by referring to the classical Poincaré equation. Properties of the Poincaré equation have been used in [69] to study the analytic properties of the zeta function of the Laplace operator given by the diffusion on certain self-similar fractals. For more details we refer to Section 4.4.

Remark 3.1 *The Sierpiński carpet is a typical example of an infinitely ramified fractal. In [70] approximations by graphs are used to define a diffusion on this fractal. By the infinite ramification, this approach is more intricate than the procedure described here. By the very recent result on the uniqueness of Brownian motion on the Sierpiński carpet (cf. [38]) this yields the same process as constructed by rescaling the classical Brownian motion restricted to finite approximations of the Sierpiński carpet in [19, 37]*

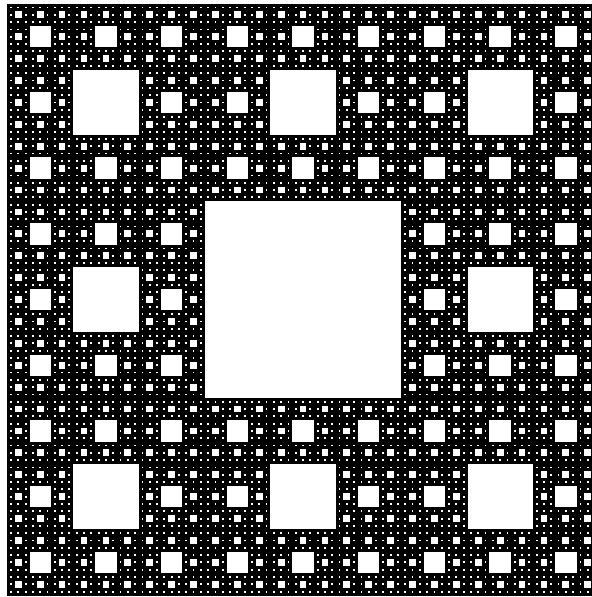


Figure 4. The Sierpiński carpet

3.3. The Laplace operator as the infinitesimal generator of a diffusion

Given a diffusion process $(X_t)_{t \geq 0}$ on a fractal K we can now define a corresponding Laplace operator. At first we define a semi-group of operators A_t by

$$A_t f(x) = \mathbb{E}_x f(X_t)$$

for functions $f \in L^2(K)$. The semi-group property

$$A_s A_t = A_{s+t}$$

of the operators comes from the Markov property of the underlying stochastic process X_t .

By [58, Chapter IX] this semi-group has an infinitesimal generator given as

$$\Delta f = \lim_{t \rightarrow 0+} \frac{A_t f - f}{t}.$$

This limit exists on a dense subspace \mathcal{F} of $L^2(K)$ and is called Laplace operator on K . This name comes from the fact that for the usual Brownian motion on a manifold this procedure yields the classical Laplace-Beltrami operator. The function $u(x, t) = A_t f(x)$ satisfies the heat equation

$$\Delta u = \partial_t u, \quad u(x, 0) = f(x).$$

It was observed in the early beginnings of the development of the theory of diffusion of fractals that the domain of Δ does not contain the restriction of any non-constant differentiable function (cf. [71]).

3.4. The Laplace operator as limit of difference operators on graphs

A totally different and more direct approach to the Laplace operator on self-similar fractals has been given by Kigami in [13]. The operator Δ is approximated by difference operators on the approximating graphs G_n . The graph Laplacians are given by

$$\Delta_n f(x) = \sum_{y \sim x, y \in G_n} p(x, y) f(y) - f(x)$$

as a weighted sum over the neighbours of x in G_n ($y \sim x$ describes the neighbourhood relation in the graph G_n). In order to make this construction compliant with the approach via stochastic processes, these operators have to be rescaled appropriately. The correct rescaling is then given by the time scaling factor λ introduced before, namely

$$\Delta f(x) = \lim_{n \rightarrow \infty} \lambda^n \Delta_n f(x).$$

3.5. Laplace operators via Dirichlet forms

Following the exposition in [16, Chapters 2 and 3] we define a sequence of quadratic forms on the finite sets V_m given in Definition 2.

Definition 4 *Let V be a finite set. Then a bilinear form \mathcal{E} on $\ell(V)$, the real functions on V is called a Dirichlet form, if the following conditions hold:*

- (i) $\forall u \in \ell(V) : \mathcal{E}(u, u) \geq 0$
- (ii) $\mathcal{E}(u, u) = 0$ implies that u is constant on V
- (iii) for $u \in \ell(V)$ and $\bar{u}(x) = \max(0, \min(u(x), 1))$ the inequality $\mathcal{E}(u, u) \geq \mathcal{E}(\bar{u}, \bar{u})$ holds.

Definition 5 Let \mathcal{E} be a Dirichlet form on the finite set V and let U be a proper subset of V . Then the restriction of \mathcal{E} to U is defined as

$$\mathcal{R}_{V,U}(\mathcal{E})(u, u) = \min \{ \mathcal{E}(v, v) \mid v \in \ell(V), v|_U = u \}. \quad (16)$$

On the level of the coefficient matrices of the Dirichlet forms, the operation of restriction is given in terms of the Schur complement.

Definition 6 Let $(V_n, \mathcal{E}_n)_n$ be a sequence of increasing finite sets V_n and Dirichlet forms \mathcal{E}_n on V_n . The sequence is called compatible, if

$$\mathcal{R}_{V_{n+1}, V_n}(\mathcal{E}_{n+1}) = \mathcal{E}_n$$

holds for all n .

For a compatible sequence $(V_n, \mathcal{E}_n)_n$ and a function f on K , the sequence $(\mathcal{E}_n(f|_{V_n}, f|_{V_n}))_n$ is increasing by definition and thus converges to a value in $[0, \infty]$. This makes the following definition natural.

Definition 7 Let $(V_n, \mathcal{E}_n)_n$ be a compatible sequence of Dirichlet forms. Let

$$\mathcal{D} = \left\{ f : K \rightarrow \mathbb{R} \mid \lim_{n \rightarrow \infty} \mathcal{E}_n(f|_{V_n}, f|_{V_n}) < \infty \right\}.$$

Then for all $f \in \mathcal{D}$

$$\mathcal{E}(f, f) = \lim_{n \rightarrow \infty} \mathcal{E}_n(f|_{V_n}, f|_{V_n})$$

defines a Dirichlet form on K , and \mathcal{D} is its domain.

In order to make the sequence of Dirichlet forms coherent with the self-similar structure of K , we require the following self-similarity condition for \mathcal{E}_n

$$\mathcal{E}_{n+1}(f, f) = \lambda \sum_{i=1}^m r_i^{-1} \mathcal{E}_n(f \circ F_i, f \circ F_i),$$

where r_i ($i = 1, \dots, m$) are positive weights and λ is a proportionality factor. Furthermore, the sequence of forms has to be compatible, which amounts to the equation

$$\lambda \mathcal{R}_{V_1, V_0} \left(\sum_{i=1}^m r_i^{-1} \mathcal{E}_0(\cdot \circ F_i, \cdot \circ F_i) \right) = \mathcal{E}_0(\cdot, \cdot), \quad (17)$$

which comes as a solution of a non-linear eigenvalue equation. This equation plays the same role in the Dirichlet form approach as the equations (14) play for the approach via random walks. The Dirichlet form \mathcal{E} on K is then defined as

$$\mathcal{E}(f, f) = \lim_{n \rightarrow \infty} \lambda^n \sum_{w \in S^n} r_w^{-1} \mathcal{E}_0(f \circ F_w|_{V_0}, f \circ F_w|_{V_0}),$$

where $S = \{1, \dots, m\}$, $r_w = r_{w_1} \cdots r_{w_n}$ for $w = w_1 \dots w_n$ and $F_w = F_{w_1} \circ \dots \circ F_{w_n}$.

Remark 3.2 There are some additional technical problems concerning this construction of Dirichlet forms, which arise from the fact that in general the form is supported only on a proper subset of K . In [16, Chapter 3] sufficient conditions for the weights r_i and the form \mathcal{E}_0 are given, which ensure that the form \mathcal{E} is supported on the whole set K .

Given a Dirichlet form \mathcal{E} together with its domain \mathcal{D} and a measure μ on K , we can now define the associated Laplace operator on K by

$$\forall v \in \mathcal{D} \cap L^2(\mu) : \mathcal{E}(u, v) = -\langle \Delta_\mu u, v \rangle_{L^2(\mu)}, \quad (18)$$

which defines Δ_μ , if \mathcal{D} is dense in $L^2(\mu)$. Notice that this is the same equation as in the manifold case (11).

In the case of a self-similar fractal K as described in Section 2 the “natural” measure on K is the according Hausdorff measure \mathcal{H}_K^ρ . In this case we omit the subscript μ .

For more information on Dirichlet forms in general and their applications to the description of diffusion processes, we refer to the monograph [57]. For the specifics of Dirichlet forms on fractals we refer to [16, 72].

4. Spectral analysis on fractals

4.1. Spectral analysis on manifolds

Let us start with a short discussion of the manifold case, which is again somehow complementary to the fractal case. As described in Section 3 the Laplace operator on a manifold M is defined via a Riemannian metric g . If M is compact with smooth (or empty) boundary (for simplicity) the Laplace operator has pure point spectrum. The eigenvalues $-\lambda_k$ are all real (by self-adjointness) and non-positive (by definition (10)). Denote the normalised eigenfunctions by ψ_k . Then the heat kernel can be written as

$$p_t(x, y) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \psi_k(x) \psi_k(y). \quad (19)$$

This expression yields the trace of the heat kernel

$$K(t) = \int_M p_t(x, x) \, d \, \text{vol}(x) = \sum_{k=0}^{\infty} e^{-\lambda_k t}. \quad (20)$$

On the other hand, if M is closed, the asymptotic behaviour of the heat kernel for $t \rightarrow 0+$ can be described very precisely

$$p_t(x, y) = (4\pi t)^{-\frac{n}{2}} e^{-d(x,y)^2/(4t)} \sum_{\ell=0}^{\infty} t^\ell a_\ell(x, y), \quad (21)$$

where the functions a_ℓ can be computed iteratively by solving certain second order partial differential equations (cf. [73, 74]). In particular, $a_0(x, y) = 1$. The values $a_\ell(x, x)$ can be expressed in terms of the functions $\Delta_y^k d(x, y)^{2m} \big|_{y=x}$ (cf. [75]). In the case of manifolds M with smooth boundary, further terms involving half integer powers of t occur

$$p_t(x, x) = t^{-\frac{n}{2}} \sum_{\ell=0}^{\infty} t^\ell a_\ell(x, x) + t^{-\frac{n}{2}} \sum_{\ell=1}^{\infty} t^{\frac{\ell}{2}} b_\ell(x, x); \quad (22)$$

the terms b_ℓ encode curvature information of the boundary ∂M . For more details we refer to [76].

Especially, this gives an asymptotic expansion of $K(t)$ (in the case of a closed manifold)

$$K(t) = (4\pi t)^{-\frac{n}{2}} \sum_{\ell=0}^{\infty} t^{\ell} \int_M a_{\ell}(x, x) \, d \operatorname{vol}(x). \quad (23)$$

From the first order asymptotic relation $K(t) \sim \operatorname{vol}(M)/(4\pi t)^{\frac{n}{2}}$ the asymptotic behaviour of the counting function

$$N(x) = \sum_{\lambda_k < x} 1 \quad (24)$$

can be obtained by a Tauberian argument giving the classical Weyl asymptotic relation

$$N(x) = \frac{\operatorname{vol}(M)}{B_n} x^{\frac{n}{2}} + o(x^{\frac{n}{2}}), \quad (25)$$

where B_n denotes the volume of the n -dimensional unit ball. The eigenvalues of Δ constitute the frequencies of the oscillations in the solutions of the wave equation $\Delta u = u_{tt}$. This led to M. Kac's famous question "Can one hear the shape of a drum?" (cf. [77]).

The precise order of magnitude of the error term has been determined as $x^{\frac{n-1}{2}}$ in the case of a smooth boundary of M in [78]. In the case of fractal boundary, this is subject of the Weyl-Berry conjecture stated in [79,80]. The original conjecture estimated the error term by $\mathcal{O}(x^{\frac{d_H(\partial M)}{2}})$, where $d_H(\partial M)$ denotes the Hausdorff dimension of the boundary of M . This was shown to be false in [81]. The Hausdorff dimension was then replaced by the Minkowski dimension $d_M(\partial M)$. It was shown in [82] that the error term is $\mathcal{O}(x^{\frac{d}{2}})$ for any $d > d_M(\partial M)$. Furthermore, it was proved in [83] that the modified conjecture is true for dimension $n = 1$; in [84] counterexamples for dimension $n > 1$ are presented. In these counterexamples disconnected manifolds are considered, thus the conjecture could still hold for connected manifolds M with fractal boundary.

More precise information on the eigenvalues is contained in the spectral zeta function

$$\zeta_{\Delta}(s) = \sum_{\lambda_k \neq 0} (\lambda_k)^{-s}, \quad (26)$$

the Dirichlet generating function of $(\lambda_k)_k$. The zeta function is connected to the trace of the heat-kernel by a Mellin transform

$$\zeta_{\Delta}(s)\Gamma(s) = \int_0^{\infty} K(t)t^{s-1} \, dt, \quad \text{for } \Re s > \frac{n}{2}. \quad (27)$$

The right hand side has an analytic continuation to the whole complex plane, which can be found by using the asymptotic expansion (23). Furthermore, (in the case of a closed manifold) the right hand side has at worst simple poles at the points $s = \frac{n}{2} - \ell$ (with $\ell \in \{0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor$ for even n and $\ell \in \mathbb{N}_0$ for odd n) with residues

$$\operatorname{Res}_{s=\frac{n}{2}-\ell} \zeta_{\Delta}(s) = \frac{1}{\Gamma(\frac{n}{2}-\ell)} \int_M a_{\ell}(x, x) \, d \operatorname{vol}(x)$$

and special values (for $\ell \in \mathbb{N}$)

$$\zeta_{\Delta}(-\ell) = \begin{cases} 0 & \text{for } n \text{ odd} \\ (-1)^{\ell} \ell! \int_M a_{\ell}(x, x) \, d \operatorname{vol}(x) & \text{for } n \text{ even} \end{cases}$$

and

$$\zeta_{\Delta}(0) = \begin{cases} -\dim \ker \Delta & \text{for } n \text{ odd} \\ \int_M a_{n/2}(x, x) \, d \operatorname{vol}(x) - \dim \ker \Delta & \text{for } n \text{ even} \end{cases}$$

(cf. [73]). The special value $\zeta'_{\Delta}(0)$ can be interpreted as the negative logarithm of the determinant of Δ . In the case of a manifold M with smooth boundary a similar reasoning gives simple poles of the zeta function at the negative half-integers $-\frac{1}{2}, -\frac{3}{2}, \dots$ and at the points $\frac{n}{2} - \frac{1}{2}, \frac{n}{2} - \frac{3}{2}, \dots, \frac{3-(-1)^n}{4}$ with residues depending on the functions b_{ℓ} .

In the case of a fractal, the above approach to the spectral zeta function, its analytic continuation, and its finer properties can not be used. The reason for this is that no asymptotic expansion of the heat kernel is known in the fractal case. In Section 4.5 we comment on the known upper and lower estimates for the heat kernel.

On the other hand, for fractals having spectral decimation, the eigenvalues can be described very precisely. In this case they turn out to constitute a finite union of level sets of solutions of the classical Poincaré functional equation. This allows then to obtain the analytic continuation of the spectral zeta function from the precise knowledge of the asymptotic behaviour of the Poincaré function; the asymptotic expansion of the trace of the heat kernel can then be obtained from the poles of the zeta function, reversing the argument in (27). The eigenvalue counting function can then be related to the harmonic measure on the Julia set of a the polynomial governing the spectral decimation.

4.2. Spectral decimation

It has been first observed by Fukushima and Shima [17, 20, 21] that the eigenvalues of the Laplacian on the Sierpiński gasket and its higher dimensional analogues exhibit the phenomenon of *spectral decimation*. Later on, spectral decimation for more general fractals has been studied by Malozemov, Strichartz, and Teplyaev [22, 23, 85].

Definition 8 (Spectral decimation) *The Laplace operator on a p. c. f. self-similar fractal G admits spectral decimation, if there exists a rational function R , a finite set A and a constant $\lambda > 1$ such that all eigenvalues of Δ can be written in the form*

$$\lambda^m \lim_{n \rightarrow \infty} \lambda^n R^{(-n)}(\{w\}), \quad w \in A, m \in \mathbb{N} \quad (28)$$

where the preimages of w under n -fold iteration of R have to be chosen such that the limit exists. Furthermore, the multiplicities $\beta_m(w)$ of the eigenvalues depend only on w and m , and the generating functions of the multiplicities are rational.

The fact that all eigenvalues of Δ are negative real implies that the Julia set of R has to be contained in the negative real axis. We will exploit this fact later.

In many cases such as the higher dimensional Sierpiński gaskets, the rational function R is conjugate to a polynomial. The method for meromorphic continuation of ζ_Δ given in Section 4.4 makes use of this assumption. Recently, Teplyaev [85] showed under the same assumption that the zeta function of the Laplacian admits a meromorphic continuation to $\Re s > -\varepsilon$ for some $\varepsilon > 0$ depending on properties of the Julia set of the polynomial given by spectral decimation. His method uses ideas similar to those used in [67] for the meromorphic continuation of a Dirichlet series attached to a polynomial. Complementary to the ideas used here, Teplyaev's method carries over to rational functions R .

4.3. Eigenvalue counting

As in the euclidean case the eigenvalue counting function

$$N(x) = \sum_{\lambda_k < x} 1$$

measures the number of eigenvalues less than x . In a physical context this quantity is referred to as the “integrated density of states”. It turns out that $N(x)$ in many cases does not exhibit a pure power law as in the euclidean case, but shows periodic fluctuations. One source of this periodicity phenomenon is actually spectral decimation, especially the high multiplicities of eigenvalues, as will become clear in Section 4.4.

Recall the definition of harmonic measure on the Julia set of a polynomial p of degree d (cf. [86]): the sequence of measures

$$\mu_n = \frac{1}{d^n} \sum_{p^{(n)}(x)=\xi} \delta_x$$

converges weakly to a limiting measure μ , the harmonic measure on the Julia set of p . The point ξ can be chosen arbitrarily.

Assume now that the Laplace operator on the fractal K admits spectral decimation with a polynomial p . Then the relation $\lim_{n \rightarrow \infty} \lambda^n p^{(-n)}(\{w\}) \in B(0, x)$ can be translated into

$$\lim_{n \rightarrow \infty} p^{(n)}(\lambda^{-n} z) = w \text{ and } |z| < x.$$

Here $B(0, x)$ denotes the ball of radius x around 0. By general facts about polynomial iteration (cf. [26]), the limit exists and defines an entire function of z , the Poincaré function $\Phi(z)$. The number of eigenvalues with $m = 0$ in $B(0, x)$ is the equal to

$$N(x) = \sum_{w \in A} N_w(x)$$

with

$$N_w(x) = \# \{z \in B(0, x) \mid \Phi(z) = w\}.$$

The following relation can be obtained from the definition of harmonic measure

$$\lim_{n \rightarrow \infty} d^{-n} N_w(\lambda^n x) = \mu(\Phi^{-1}(B(0, x)));$$

this holds for all x small enough to ensure the existence of the inverse function Φ^{-1} on $B(0, x)$.

In [87, Theorem 5.2] we could prove a relation between the asymptotic behaviour of the partial counting functions $N_w(x)$ and the harmonic measure of balls $\mu(B(0, x))$. The existence of the two limits ($\rho = \log_\lambda d$)

$$\lim_{x \rightarrow \infty} x^{-\rho} N_w(x) = \lim_{t \rightarrow 0} t^{-\rho} \mu(B(0, t))$$

is equivalent. We conjectured there, that these limits can only exist, if p is either a Chebyshev polynomial or a monomial. These are the only cases of polynomials with smooth Julia sets (cf. [88]).

Summing up the above discussion, the eigenvalue counting function can be written as

$$N(x) = \sum_{w \in A} \sum_{m=0}^{\infty} \beta_m(w) N_w(\lambda^{-m} x). \quad (29)$$

Notice, that for fixed x , these sums are actually finite. In the known cases, such as the Sierpiński gasket, the growth of $\beta_m(w)$ is stronger than d^m , which implies that the terms for large m (with $N_w(\lambda^{-m} x)$ still positive) become dominant in this sum. This shows that multiplicity of the eigenvalues has the main influence on the asymptotic behaviour of $N(x)$. Furthermore, this explains the presence of an oscillating factor in the asymptotic main term of $N(x)$. We will discuss that in more detail in Section 4.4.

4.4. Spectral zeta functions

As in the euclidean case, the eigenvalues of the Laplace operator Δ can be put into a Dirichlet generating function. This will later allow to use methods and ideas from analytic number theory to obtain more precise asymptotic information on $N(x)$. The zeta function is again given by

$$\zeta_\Delta(s) = \sum_{\lambda_k \neq 0} \lambda_k^{-s},$$

where all eigenvalues are counted with their multiplicity. The zeta function is related to the eigenvalue counting function by

$$\zeta_\Delta(s) = \int_0^\infty x^{-s} dN(x) = s \int_0^\infty N(x) x^{-s-1} dx.$$

The second relation identifies ζ_Δ as the Mellin transform of the counting function $N(x)$.

In the sequel we will exploit the consequences of spectral decimation. Not too surprisingly after Definition 8, iteration of polynomials will play an important role in this discussion. Furthermore, since the relation (28) can be expressed in terms of the Poincaré function f , properties of this function will be used to derive the meromorphic continuation of ζ_Δ to the whole complex plane.

Under the assumptions of spectral decimation, the Julia set of the polynomial p is a subset of the non-positive reals, which contains 0. By [87, Theorem 4.1] this implies that $\lambda = p'(0) \leq d^2$. By [87, Theorem 4.1] equality can only occur, if p is a Chebyshev polynomial, which would correspond to spectral decimation on the unit interval (viewed as a self-similar fractal). Thus in the cases of interest, we have that $\rho = \log_\lambda d < \frac{1}{2}$. The Poincaré function Φ is then an entire function of order ρ .

In order to find the analytic continuation of $\zeta_\Delta(s)$ to the whole complex plane, we analyse the partial zeta functions

$$\zeta_{\Phi,w}(s) = \sum_{\Phi(-\mu)=w, \mu \neq 0} \mu^{-s}. \quad (30)$$

Since $\Phi_w = 1 - \frac{1}{w}\Phi$ is a function of order $\rho = \log_\lambda d < \frac{1}{2}$, it can be expressed as a Hadamard product

$$1 - \frac{1}{w}\Phi(z) = \prod_{\Phi(-\mu)=w} \left(1 + \frac{z}{\mu}\right);$$

for $w = 0$ we have the slightly modified expression

$$\Phi_0(z) = \frac{1}{z}\Phi(z) = \prod_{\Phi(-\mu)=0, \mu \neq 0} \left(1 + \frac{z}{\mu}\right).$$

Taking the Mellin-transform of $\log \Phi_w$, which exists for $-1 < \Re s < -\rho$, we get

$$M_w(s) = \int_0^\infty (\log \Phi_w(x)) x^{s-1} dx = \frac{\pi}{s \sin \pi s} \zeta_{\Phi,w}(-s). \quad (31)$$

Thus for finding the analytic continuation of $\zeta_{\Phi,w}(s)$ to the left of its abscissa of convergence, it suffices to find the analytic continuation of $M_w(s)$ for $\Re s > -\rho$. Following slightly different lines as in [69], we consider the function

$$\Psi_w(z) = \frac{p(\Phi(z)) - w}{a_d(\Phi(z) - w)^d} = \frac{\Phi_w(\lambda z)}{a_d(-w)^{d-1} \Phi_w(z)^d}.$$

Then we have

$$\log \Psi_w(z) = \log \Phi_w(\lambda z) - d \log \Phi_w(z) - \log a_d - (d-1) \log(-w)$$

and this function tends to 0 exponentially for $z \rightarrow +\infty$. Taking the Mellin transform we obtain

$$\begin{aligned} & (\lambda^s - d)M_w(s) \\ &= \int_0^\infty (\log \Phi_w(\lambda x) - d \log \Phi_w(x)) x^{s-1} dx \\ & \quad \text{for } -1 < \Re s < -\rho \\ &= \int_0^1 (\log \Phi_w(\lambda x) - d \log \Phi_w(x) - \log a_d - (d-1) \log(-w)) x^{s-1} dx \\ & \quad + (\log a_d + (d-1) \log(-w)) \frac{1}{s} + \int_1^\infty (\log \Phi_w(\lambda x) - d \log \Phi_w(x)) x^{s-1} dx \\ & \quad \text{for } \Re s > -1 \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 (\log \Psi_w(x)) x^{s-1} dx \\
&+ \int_1^\infty (\log \Phi_w(\lambda x) - d \log \Phi_w(x) - \log a_d - (d-1) \log(-w)) x^{s-1} dx \\
&\quad \text{for } \Re s > 0 \\
&= \int_0^\infty (\log \Psi_w(x)) x^{s-1} dx.
\end{aligned}$$

Reading the fourth line of this computation shows that $M_w(s)$ has a simple pole at $s = 0$ with residue

$$\operatorname{Res}_{s=0} M_w(s) = \frac{\log a_d}{d-1} + \log(-w).$$

Furthermore, this computation shows that $M_w(s)$ is holomorphic in the half-plane $\Re s > 0$. Using (31) gives the analytic continuation of $\zeta_{\Phi,w}(s)$ for $\Re s < 0$

$$\zeta_{\Phi,w}(s) = \frac{\lambda^s s \sin \pi s}{\pi(1 - d\lambda^s)} \int_0^\infty (\log \Psi_w(x)) x^{-s-1} dx. \quad (32)$$

This shows that $\zeta_{\Phi,w}(-m) = 0$ for $m \in \mathbb{N}_0$ (for $s = 0$ the double zero of $s \sin \pi s$ cancels the simple pole of $M_w(-s)$). These could be called the “trivial zeros” as in the case of the Riemann zeta function. Furthermore, we obtain

$$\zeta'_{\Phi,w}(0) = \frac{\log a_d}{d-1} + \log(-w).$$

The equation (32) even lends itself to the numerical computation of values of $\zeta_{\Phi,w}(s)$ for $\Re s < 0$, as we will see in Section 4.5.2.

By our assumption on spectral decimation, the generating functions of the multiplicities of the eigenvalues are rational

$$R_w(z) = \sum_{m=0}^{\infty} \beta_m(w) z^m.$$

Thus we can write the spectral zeta function of Δ as

$$\zeta_\Delta(s) = \sum_{w \in A} \sum_{m=0}^{\infty} \beta_m(w) \sum_{\Phi(-\mu)=w} (\lambda^m \mu)^{-s} = \sum_{w \in A} R_w(\lambda^{-s}) \zeta_{\Phi,w}(s). \quad (33)$$

Since all functions involved in the last (finite) sum are meromorphic in the whole complex plane, we have found the meromorphic continuation of ζ_Δ to the whole complex plane. The functions $\zeta_{\Phi,w}(s)$ have only simple poles in the points $s = \log_\lambda d + \frac{2k\pi i}{\log \lambda}$ ($k \in \mathbb{Z}$). Furthermore, the residues of these poles do not depend on w . All other poles of ζ_Δ come from the poles of the functions $R_w(\lambda^{-s})$. Since these are rational functions of λ^{-s} , their poles are equally spaced on vertical lines.

Summing up, $\zeta_\Delta(s)$ has a meromorphic continuation to the whole complex plane with poles in the points

$$s = -\log_\lambda \beta_{w,j} + \frac{2k\pi i}{\log \lambda} \text{ with } k \in \mathbb{Z},$$

where the $\beta_{w,j}$ are the poles of the rational functions $R_w(z)$ and (at most) simple poles in the points

$$s = \log_\lambda d + \frac{2k\pi i}{\log \lambda} \text{ with } k \in \mathbb{Z}.$$

Furthermore, since the functions $\zeta_{\Phi,w}(s)$ are bounded for $\Re s \geq \log_\lambda d + \varepsilon$ and the functions $R_w(\lambda^{-s})$ are bounded along every vertical line, which contains no poles, the function $\zeta_\Delta(s)$ is bounded along every vertical line $c + it$ for $c > \log_\lambda d$, which does not contain a pole of any of the functions $R_w(\lambda^{-s})$.

In the case of the Sierpiński gasket and its higher dimensional analogues, the rightmost poles of ζ_Δ come from the rational functions $R_w(\lambda^{-s})$. Furthermore, the relation

$$\sum_{w \in A} R_w(1/d) = 0$$

holds, which amounts in a (still mysterious) cancellation of the poles of the functions $\zeta_{\Phi,w}$ in (33). Thus, the analytic behaviour of the function ζ_Δ is mainly governed by the functions R_w . This means that in this respect the effects of multiplicity prevail over the individual eigenvalues.

We now investigate the asymptotic behaviour of $N(x)$ under the assumption that $\beta_m(w)$ grows exponentially faster than d^m for some $w \in A$. This implies that the corresponding term $R_s(\lambda^{-s})$ has poles to the right of $\Re s = \log_\lambda d$. We use the classical Mellin-Perron formula (cf. [89]) to express $N(x)$ in terms of $\zeta_\Delta(s)$

$$N(x) = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta_\Delta(s) x^s \frac{ds}{s},$$

for any c such that $R_w(\lambda^{-s})$ has no poles in the half-plane $\Re(s) \geq c$. Now the line of integration can be shifted to the left to $\Re(s) = c'$ for $\lambda^{c'} > d$ but such that at least one of the functions $R_w(\lambda^{-s})$ has poles to the right of c' . This is justified, because $\zeta_\Delta(\sigma + it)$ remains bounded for $|t| \rightarrow \infty$ and $\sigma \geq c'$. Then we have

$$N(x) = \lim_{T \rightarrow \infty} \left(\frac{1}{2\pi i} \int_{c'-iT}^{c'+iT} \zeta_\Delta(s) x^s \frac{ds}{s} - \sum_{2\pi|k| < T \log \lambda} \operatorname{Res}_{s=d_S/2+2k\pi i/\log \lambda} \frac{\zeta_\Delta(s) x^s}{s} \right),$$

where we denote the real part of the poles of $R_w(\lambda^{-s})$ by $d_S/2$ (the “spectral dimension”). Now the limit of the integral $\int_{c'-iT}^{c'+iT}$ can be shown to exist for $T \rightarrow \infty$, which shows that the limit

$$\lim_{T \rightarrow \infty} \sum_{2\pi|k| < T \log \lambda} \operatorname{Res}_{s=d_S/2+2k\pi i/\log \lambda} \frac{\zeta_\Delta(s) x^s}{s}$$

exists. This can be rewritten as $x^{d_S/2} H(\log_\lambda x)$ for a periodic continuous function H given by its Fourier expansion. The limit of the integral can be shown to be $\mathcal{O}(x^{c'})$. Thus we have shown

$$N(x) = x^{d_S/2} H(\log_\lambda x) + \mathcal{O}(x^{c'}). \quad (34)$$

Especially, the limit $\lim_{x \rightarrow \infty} x^{-d_S/2} N(x)$ does not exist.

We remark here that there exists totally different approach to zeta functions of fractals due to M. Lapidus and his collaborators [28, 29, 90, 91]. In this geometric approach the volume of tubular neighbourhoods of the fractal G

$$V_G(\varepsilon) = \text{vol}(\{x \in \mathbb{R}^n \mid d(x, G) < \varepsilon\})$$

is analysed. The asymptotic behaviour of $V_G(\varepsilon)$ for $\varepsilon \rightarrow 0$ gives rise to the definition of a zeta function. In this geometric context the complex solutions of (4) occur as poles of the zeta function; they are called the “complex dimensions” of the fractal G in this context. This approach is motivated by the definition of Minkowski content, which itself turned out to be too restrictive to measure (most of the) self-similar fractals.

4.5. Trace of the heat kernel

The diffusion semi group A_t introduced in Section 3.3 can be given in terms of the heat kernel $p_t(x, y)$. As opposed to the situation in the euclidean case, the knowledge on behaviour of the heat kernel for $t \rightarrow 0$ is by far not as precise. Kumagai [92] proved the following lower and upper estimates of the form

$$t^{-\frac{d_S}{2}} \exp \left(-c_1 \left(\frac{d(x, y)^{d_w}}{t} \right)^{\frac{1}{d_w-1}} \right) \lesssim p_t(x, y) \quad (35)$$

$$p_t(x, y) \lesssim t^{-\frac{d_S}{2}} \exp \left(-c_2 \left(\frac{d(x, y)^{d_w}}{t} \right)^{\frac{1}{d_w-1}} \right) \quad (36)$$

where d_S and d_w are the spectral and the walk dimension of the fractal, respectively. These dimensions are related to the Hausdorff dimension d_f of the fractal via the so called *Einstein relation* $d_S d_w = 2d_f$. In the fractal case usually $d_w > 2$, as opposed to the euclidean case, where $d_w = 2$, which implies $d_S = d_f$.

Only recently, a conjecture by Barlow and Perkins [8] could be proved by Kajino [93]. Namely that for any $x \in K$ the limit

$$\lim_{t \rightarrow 0} t^{\frac{d_S}{2}} p_t(x, x)$$

does *not* exist for a large class of self-similar fractals. This gives an indication, why obtaining more precise information on the heat kernel than the estimates (35) and (36) would be very difficult.

Even if the precise behaviour of the heat kernel seems to be far out of reach, the trace of the heat kernel

$$K(t) = \int_K p_t(x, x) d\mathcal{H}(x)$$

can still be analysed in some detail under the assumption of spectral decimation. The reason for this are the two relations between $K(t)$, $N(x)$, and ζ_Δ

$$K(t) = \int_0^\infty e^{-tx} dN(x) = t \int_0^\infty N(x) e^{-xt} dx \quad (37)$$

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^\infty K(t) t^{s-1} dt. \quad (38)$$

The first expresses $K(t)$ as the Laplace transform of $N(x)$, the second gives ζ_Δ as Mellin transform of $K(t)$.

Given the precise knowledge on the zeta function obtained in Section 4.4, we can use the Mellin inversion formula to derive asymptotic information on $K(t)$ for $t \rightarrow 0$. We have for $c > \frac{d_S}{2}$

$$K(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta_\Delta(s) \Gamma(s) t^{-s} ds, \quad (39)$$

where integration is along the vertical line $\Re s = c$. We notice that the Gamma function decays exponentially along vertical lines, whereas Dirichlet series grow at most polynomially by the general theory of Dirichlet series (cf. [94]). Thus convergence of the integral is guaranteed.

Shifting the line of integration in (39) to $\Re s = -M - \frac{1}{2}$ ($M \in \mathbb{N}$) and taking the poles of the integrand into account, we obtain for $t \rightarrow 0$

$$K(t) = t^{-\frac{d_S}{2}} H(\log_\lambda t) + \sum_j t^{-\alpha_j} H_j(\log_\lambda t) + \mathcal{O}(t^{M+\frac{1}{2}}). \quad (40)$$

Here, H and H_j are periodic continuous functions of period 1, whose Fourier coefficients are given as the residues of $\zeta_\Delta(s) \Gamma(s)$ in the poles on the lines $\Re s = \frac{d_S}{2}$ or $\Re s = \alpha_j$ respectively. By the strong decay of the Gamma function, these functions are even real analytic. The values $\alpha_j + 2k\pi i / \log \lambda$ come as the poles of the functions $R_w(\lambda^{-s})$. This shows that there are only finitely many α_j . Since M can be made arbitrarily large, the error term decays faster than any positive power of t for $t \rightarrow 0$.

The existence of complex poles of the zeta function thus implies the presence of periodically oscillating terms in the asymptotic behaviour of the trace of the heat kernel for $t \rightarrow 0$. This implies that the limit $\lim_{t \rightarrow 0} t^{d_S/2} p_t(x, x)$ does not exist on a set of positive measure for x in accordance with the above mentioned result by N. Kajino [93]. The consequences of the presence of complex poles of the zeta function to properties of the heat kernel, the density of states, and the partition function as well as the physical implications of the resulting fluctuating behaviour in these quantities have been discussed in [30, 31].

4.5.1. Casimir energy on fractals As an application of the spectral zeta function and its properties we compute the Casimir energy of a fractal. We follow the lines of the exposition in [76] and refer to this book for further details.

Consider the differential operator $P = -\frac{\partial^2}{\partial \tau^2} + \Delta$ on $(\mathbb{R}/\frac{1}{\beta}\mathbb{Z}) \times G$. As usual the parameter $\beta = 1/kT$. The zeta function of the operator P is then given by

$$\zeta_P(s) = \frac{1}{\Gamma(s)} \int_0^\infty K(t) \sum_{n \in \mathbb{Z}} e^{-\frac{4\pi^2 n^2}{\beta^2} t} t^{s-1} dt.$$

Using the theta function relation

$$\sum_{n \in \mathbb{Z}} e^{-\frac{4\pi^2 n^2}{\beta^2} t} = \frac{\beta}{2\sqrt{\pi t}} \sum_{n \in \mathbb{Z}} e^{-\frac{\beta^2 n^2}{4t}}$$

we obtain

$$\zeta_P(s) = \frac{\beta}{2\sqrt{\pi}\Gamma(s)}\Gamma\left(s - \frac{1}{2}\right)\zeta_\Delta\left(s - \frac{1}{2}\right) + \frac{\beta}{\sqrt{\pi}\Gamma(s)}\int_0^\infty K(t)\sum_{n=1}^\infty e^{-\frac{\beta^2 n^2}{4t}}t^{s-\frac{3}{2}}dt. \quad (41)$$

The derivative $\zeta'_P(0)$ can be interpreted as $-\log \det P$, the logarithm of the (regularised) determinant of P .

For the rescaled operator P/μ^2 we have

$$\zeta'_{P/\mu^2}(0) = \zeta'_P(0) + \zeta_P(0) \ln \mu^2.$$

From (41) we compute

$$\zeta_P(0) = -\beta \operatorname{Res}_{s=-\frac{1}{2}} \zeta_\Delta(s),$$

which vanishes in the case of the Sierpiński gasket, because ζ_Δ does not have a pole at $-\frac{1}{2}$. Furthermore, we get

$$\zeta'_P(0) = -\beta\zeta_\Delta\left(-\frac{1}{2}\right) + \frac{\beta}{\sqrt{\pi}}\sum_{n=1}^\infty\sum_{j=1}^\infty\int_0^\infty e^{-\frac{\beta^2 n^2}{4t}-\lambda_j t}t^{-\frac{3}{2}}dt.$$

The integral and the summation over n can be evaluated explicitly, which finally gives

$$\zeta'_P(0) = -\beta\zeta_\Delta\left(-\frac{1}{2}\right) - 2\sum_{j=1}^\infty \ln\left(1 - e^{-\beta\sqrt{\lambda_j}}\right).$$

The case of the Sierpiński gasket and other self-similar fractal shows an important contrast to the manifold case, where the zeta function (generically) has a pole at $-\frac{1}{2}$ (see Section 4.1). In the manifold case the value $\zeta_\Delta(-\frac{1}{2})$ has to be replaced by the *finite part* of $\zeta_\Delta(s)$ at $s = -\frac{1}{2}$, the function minus the principal part at the polar singularity (cf. [76]).

Now the energy of the system is given by

$$E = -\frac{1}{2}\frac{\partial}{\partial\beta}\zeta'_{P/\mu^2}(0) = \frac{1}{2}\zeta_\Delta\left(-\frac{1}{2}\right) + \sum_{j=1}^\infty \frac{\sqrt{\lambda_j}}{e^{\beta\sqrt{\lambda_j}} - 1}.$$

Letting $\beta \rightarrow \infty$, which is equivalent to letting temperature tend to 0, gives the Casimir energy

$$E_{\text{Cas}} = \frac{1}{2}\zeta_\Delta\left(-\frac{1}{2}\right)$$

for fractals, whose zeta function has no pole at $-\frac{1}{2}$.

4.5.2. Numerical computations We will now describe the numerical computation of the values $\zeta_{\Phi,w}(-1/2)$, which are needed for the computation of $\zeta_\Delta(-1/2)$ in the case of the Sierpiński gasket. This fractal admits spectral decimation with the polynomial $p(x) = x(x+5)$ (cf. [3, 17]). The corresponding Poincaré function is then given by the unique holomorphic solution of the equation

$$\Phi(5z) = \Phi(z)(\Phi(z) + 5), \quad \Phi(0) = 0, \quad \Phi'(0) = 1. \quad (42)$$

We use the expression for $\zeta_\Delta(s)$ for Dirichlet boundary conditions derived in [69, Section 7]

$$\zeta_\Delta^D(s) = 5^{-s}\zeta_{\Phi,-2}(s) + \frac{3}{5^s(5^s-1)(5^s-3)}\zeta_{\Phi,-3}(s) + \frac{2 \cdot 5^s - 5}{(5^s-1)(5^s-3)}\zeta_{\Phi,-5}(s).$$

Combining this with (32) and inserting $s = -\frac{1}{2}$ gives

$$\begin{aligned} \zeta_\Delta^D\left(-\frac{1}{2}\right) &= \frac{1}{2\pi(\sqrt{5}-2)} \left[\sqrt{5} \int_0^\infty \log \Psi_{-2}(x) x^{-\frac{1}{2}} dx \right. \\ &\quad \left. + \frac{155 + 135\sqrt{5}}{88} \int_0^\infty \log \Psi_{-3}(x) x^{-\frac{1}{2}} dx - \frac{245 + 47\sqrt{5}}{124} \int_0^\infty \log \Psi_{-5}(x) x^{-\frac{1}{2}} dx \right]. \end{aligned} \quad (43)$$

From the general information on the asymptotic behaviour of Poincaré functions derived in [87, 95] we obtain estimates of the form

$$\exp(C_1 x^\rho) \leq \Phi(x) \leq \exp(C_2 x^\rho)$$

for positive constants C_1 and C_2 , $\rho = \log_5 2$ and valid for $x \geq x_0$. Such estimates can be proved easily by first showing the estimate for an interval of the form $[x_0, 5x_0]$ and then extending it by using the functional equation (42). For instance, we used $C_1 = 1$, $C_2 = 1.08$, and $x_0 = 10$. Then the functions $\log \Psi_w(x)$ tend to 0 like $\exp(-C_2 x^\rho)$ for $x \rightarrow \infty$. Given a precision goal $\varepsilon > 0$, we choose T so large that

$$\int_T^\infty \exp(-C_2 x^\rho) x^{-\frac{1}{2}} dx < \varepsilon.$$

Then the improper integrals in (43) can be replaced by \int_0^T . The functions $\Psi_w(x)$ can be computed to high precision by using the power series representation of $\Phi(x)$ for $|x| \leq 1$ and the functional equation (42) to obtain

$$\Phi(x) = p^{(k+1)}(\Phi(x/5^{k+1}))$$

for $5^k < |x| \leq 5^{k+1}$. This allows the computation of the remaining finite integrals up to precision ε . We obtained

$$E_{\text{Cas}}^D = 0.5474693544 \dots$$

for the Casimir energy of the two-dimensional Sierpiński gasket with Dirichlet boundary conditions.

Similarly, we have for Neumann boundary conditions (cf. [69])

$$\zeta_\Delta^N(s) = \frac{1}{(5^s-1)(5^s-3)} ((2 \cdot 5^s - 5)\zeta_{\Phi,-3}(s) + 5^s \zeta_{\Phi,-5}(s)).$$

This gives by the same numerical estimates as before

$$E_{\text{Cas}}^N = 2.134394089264 \dots$$

for the Casimir energy two-dimensional Sierpiński gasket with Neumann boundary conditions.

4.6. Self-similarity and the renewal equation

Let K be a post-critically finite self similar fractal with self-similar structure $(K, \Sigma, (F_i)_{i=1}^m)$ with contraction ratios α_i and Hausdorff-dimension ρ . Let K be equipped with a Dirichlet form with parameters r_i and λ as described in Section 3.5. Denote by $N_D(x)$ and $N_N(x)$ the eigenvalue counting functions for the Laplace operator under Dirichlet and Neumann boundary conditions, respectively. Then the following crucial fact was observed in [96]

$$\sum_{i=1}^m N_D\left(\frac{r_i \alpha_i^\rho}{\lambda} x\right) \leq N_D(x) \leq N_N(x) \leq \sum_{i=1}^m N_N\left(\frac{r_i \alpha_i^\rho}{\lambda} x\right), \quad (44)$$

where α_i are the contraction ratios introduced in Section 2, ρ is the Hausdorff dimension of K , and the r_i are the weights used for the construction of the Dirichlet form \mathcal{E} in Section 3.5. Furthermore, the inequalities

$$N_D(x) \leq N_N(x) \leq N_D(x) + \#(V_0)$$

hold, where V_0 is defined in Definition 2. Setting

$$\gamma_i = \left(\frac{r_i \alpha_i^\rho}{\lambda}\right)^{1/2},$$

we end up with the equation

$$N_D(x) = \sum_{i=1}^m N_D(\gamma_i^2 x) + g(x), \quad (45)$$

where $g(x)$ is defined as the difference between the left hand side and the sum on the right hand side, and remains bounded by (44). A similar equation holds for $N_N(x)$.

Equation (45) can be transformed into the classical renewal equation occurring in probability theory (cf. [97]). The asymptotic behaviour of the solutions of this equation is described by the following theorem (stated as in [16]).

Theorem 4.1 (“Renewal Theorem”) *Let $t^* > 0$ and f be a measurable function on \mathbb{R} such that $f(t) = 0$ for $t < t^*$. If f satisfies the renewal equation*

$$f(t) = \sum_{j=1}^N p_j f(t - \alpha_j) + u(t),$$

where $\alpha_1, \dots, \alpha_N$ are positive real numbers, $p_j > 0$ for $j = 1, \dots, N$ and $\sum_{j=1}^N p_j = 1$. Assume that u is non-negative and directly Riemann integrable on \mathbb{R} with $u(t) = 0$ for $t < t^$. Then the following conclusions hold:*

- (i) *arithmetic (or lattice) case: if the group generated by the α_j is discrete (i.e. there exist a $T > 0$ and integers m_j with greatest common divisor 1 such that $\alpha_j = T m_j$; all the ratios α_i/α_j are then rational), then $\lim_{t \rightarrow \infty} |f(t) - G(t)| = 0$, where the T -periodic function G is given by*

$$G(t) = \left(\sum_{j=1}^N p_j m_j\right)^{-1} \sum_{k=-\infty}^{\infty} u(t + kT).$$

(ii) *non-arithmetic (non-lattice) case: if the group generated by the α_j is dense in \mathbb{R} (i.e. at least one of the ratios α_i/α_j is irrational), then*

$$\lim_{t \rightarrow \infty} f(t) = \left(\sum_{j=1}^N p_j \alpha_j \right)^{-1} \int_{-\infty}^{\infty} u(t) dt.$$

Corollary 4.1 *Let f be a solution of the equation*

$$f(x) = \sum_{i=1}^m f(\gamma_i^2 x) + g(x),$$

where $0 < \gamma_i < 1$ and g is a bounded function. Let d_S be the unique positive solution of the equation

$$\sum_{i=1}^m \gamma_i^{d_S} = 1,$$

then the following assertions hold:

(i) *arithmetic (or lattice) case: if the group generated by the values $\log \gamma_j$ is discrete, generated by $T > 0$, then*

$$f(x) = x^{d_S/2} (G((\log x)/2) + o(1))$$

for a periodic function G of period T .

(ii) *non-arithmetic (or non-lattice) case: if the group generated by the values $\log \gamma_j$ is dense, then*

$$\lim_{x \rightarrow \infty} f(x) x^{-d_S/2}$$

exists.

The corollary is an immediate consequence of the Theorem by setting $f(e^t) = e^{d_S t/2} F(t)$ and applying the Theorem to F and $g(e^t) e^{-d_S t/2}$.

Summing up, we have the following theorem.

Theorem 4.2 ([96, Theorem 2.4]) *Let $(K, \Sigma, (F_i)_{i=1}^m)$ be a self similar structure with contraction ratios α_i and Hausdorff-dimension ρ . Let K be equipped with a Dirichlet form with parameters r_i and λ as described in Section 3.5. Let d_S be the unique positive solution of the equation*

$$\sum_{i=1}^m \left(\frac{r_i \alpha_i^\rho}{\lambda} \right)^{d_S/2} = 1,$$

the “spectral dimension” of the harmonic structure of K . Then the following assertions hold:

(i) *lattice case: if the group generated by the values $\log(r_i \alpha_i^\rho / \lambda)$ is discrete, then*

$$N_D(x) = x^{d_S/2} (G((\log(x))/2) + o(1))$$

for a periodic function G , which is non-constant in general.

(ii) *non-lattice case: if the group generated by the values $\log(r_i \alpha_i^\rho / \lambda)$ is dense, then*

$$\lim_{x \rightarrow \infty} N_D(x) x^{-d_s/2}$$

exists.

The behaviour of $N_N(x)$ for $x \rightarrow \infty$ is the same.

Remark 4.1 *Similar ideas are used in [98] to study the asymptotic expansion of the eigenvalue counting function of the Laplace operator on an open set \mathfrak{G} , which is formed from an open set G_0 with smooth boundary as*

$$\mathfrak{G} = \bigcup_{n=0}^{\infty} G_n$$

with

$$G_{n+1} = \bigcup_{i=1}^m F_i(G_n),$$

with similitudes F_i and all the unions assumed to be disjoint.

Remark 4.2 *Very recently, N. Kajino [99] could extend this approach to non-p. c. f. fractals such as the Sierpiński carpets. He observed that in order to have an inequality of the form (44) it suffices to have the corresponding self-similarity property*

$$\mathcal{E}(f, f) = \lambda \sum_{i=1}^m r_i^{-1} \mathcal{E}(f \circ F_i, f \circ F_i)$$

of the underlying Dirichlet form. This self-similarity together with symmetries characterises the Dirichlet form on the Sierpiński carpet uniquely, as was shown in [38]

5. Exploiting self-similarity

5.1. Decimation invariance and equations with rescaling

As has been described in Section 3.2, diffusion processes on decimation invariant finitely ramified fractals, belonging to a certain class, may be defined as limits of discrete simple symmetric nearest-neighbour random walks on the associated approximating graphs. The analysis is based on a renormalisation type argument, involving self-similarity and decimation invariance .

Consider, for example, the Sierpiński gasket Γ . It can be approximated by “Sierpiński lattice” graphs G_n . Let $X_t^{(n)}$ be a simple random walk on G_n . Then, according to Goldstein [6] and Kusuoka [7],

$$2^{-n} X_{[5^n t]}^{(n)} \Longrightarrow X_t$$

as $n \rightarrow \infty$, where X_t is a diffusion process on Γ . (Here, \Longrightarrow means convergence in distribution and $[\cdot]$ is the integer part function.)

X_t is a Markov process with continuous sample paths (in fact, a Feller process), which is itself invariant under the rescaling $x \rightarrow 2x$, $t \rightarrow 5t$.

As has been explained in Section 3.2, equation (15) in this instance has the form

$$\Phi(\lambda z) = R(\Phi(z)), \quad (46)$$

where $\lambda = 5$; $R(z) = \psi(z) = \frac{z^2}{4-3z}$ is a rational function of z .

Denoting here $\Phi(z) = 1/\Psi(z)$ we have also

$$\Psi(\lambda z) = P(\Psi(z)) \quad (47)$$

where $\lambda = 5$; $P(z) = 4z^2 - 3z$ is a polynomial.

Both (46) and (47) are examples of the *Poincaré equation* which will be discussed below.

5.2. Functional equations in the theory of branching processes

Iterative functional equations, and the Poincaré equation. in particular, occur also in the context of *branching processes* (cf. [9]).

Here a probability generating function

$$q(z) = \sum_{n=0}^{\infty} p_n z^n$$

encodes the offspring distribution, where $p_n \geq 0$ is the probability that an individual has n offsprings in the next generation (note that $q(1) = 1$). The growth rate $\lambda = q'(1)$ determines whether the population is increasing ($\lambda > 1$) or dying out ($\lambda \leq 1$). In the first case the branching process is called *super-critical*. The probability generating function $q^{(n)}(z)$ (n -th iterate of q) encodes the distribution of the size X_n of the n -th generation under the offspring distribution q . In the case of a super-critical branching process it is known that the random variables $\lambda^{-n} X_n$ tend to a limiting random variable X_{∞} . The moment generating function of this random variable

$$f(z) = \mathbb{E} e^{-zX_{\infty}}$$

satisfies the functional equation (cf. [9])

$$f(\lambda z) = q(f(z)), \quad (48)$$

which is yet another example of the *Poincaré equation*.

If $q(z) = \frac{1}{P(1/z)}$, where P is polynomial, then (48) coincides with (47).

In general, such a reduction to (47) is possible, when q is conjugate to a polynomial by a Möbius transformation.

5.3. Historical remarks on the Poincaré equation

In his seminal papers [100, 101]. H. Poincaré has studied the equation

$$f(\lambda z) = R(f(z)), \quad z \in \mathbf{C}, \quad (49)$$

where $R(z)$ is a rational function and $\lambda \in \mathbf{C}$. He proved that, if $R(0) = 0$, $R'(0) = \lambda$, and $|\lambda| > 1$, then there exists a meromorphic or entire solution of (49). After Poincaré, (49) is called *the Poincaré equation* and solutions of (49) are called *the Poincaré functions*

. The next important step was made by Valiron [102, 103], who investigated the case, where $R(z) = P(z)$ is a polynomial, i.e.

$$f(\lambda z) = P(f(z)), \quad z \in \mathbf{C}, \quad (50)$$

and obtained conditions for the existence of an entire solution $f(z)$. Furthermore, he derived the following asymptotic formula for $M(r) = \max_{|z| \leq r} |f(z)|$:

$$\ln M(r) \sim r^\rho Q\left(\frac{\ln r}{\ln |\lambda|}\right), \quad r \rightarrow \infty. \quad (51)$$

Here $Q(z)$ is a 1-periodic function bounded between two positive constants, $\rho = \frac{\ln m}{\ln |\lambda|}$ and $m = \deg P(z)$.

Various aspects of the Poincaré functions have been studied in the papers [87, 95, 104, 105, 106, 107, 108, 109].

5.4. Applications: diffusion on fractals

In addition to (51), in applications (see Sections 4.4 and 4.5 above, in particular) it is important to know asymptotics of entire solutions $f(z)$ in certain angular regions and even on specific rays $re^{i\vartheta}$ of the complex plane.

In the following section we present some recent results in this direction ([87, 95]).

Note, that in the aforementioned applications (diffusion on fractals) scaling factor λ is *real* and, also, polynomial $P(z)$ has *real* coefficients.

Nevertheless, we prefer to start our exposition in section 5.5 below from the general case and turn to the *real* case afterwards. We think that, these general facts on the Poincaré equation are interesting in their own, and hope that, maybe, they will also find applications in the future.

5.5. Asymptotics along spirals and asymptotics along rays

Here we intend to describe further results of Valiron's type . We start our presentation from some of our recent results [87, 95].

It turns out that asymptotic behaviour of *the Poincaré functions* heavily depends on the arithmetic nature of $\arg \lambda$, where λ is the scaling factor in (50). Denote $\arg \lambda = 2\pi\beta$.

The following statement is true, for *irrational* β ([95]) :

Theorem 5.1 *If $\arg \lambda = 2\pi\beta$ and β is irrational, then $f(z)$ is unbounded along any ray ϑ . Moreover, if we denote $\varphi(z) = \ln |f(z)|$ (where main branch of logarithm is taken) then there exists a sequence $r_n \rightarrow \infty$, such that the limit*

$$\lim_{n \rightarrow \infty} \frac{\varphi(r_n e^{i\vartheta})}{r_n^\rho} = L \quad (52)$$

exists and $L > 0$.

As far as we know this phenomenon has not been mentioned in the literature before.

On the other hand, if β is rational (and, in particular, if $\beta = 0$, i.e. λ is real) $f(z)$ may be bounded on some rays and even in whole sectors. Nevertheless, for rational β , the limit (52) still exists under some additional assumptions. Denote $\beta = t/s$ and suppose that t, s are relatively prime. Put $q = \lambda^s$ (note that $1 < q \in \mathbf{R}$).

The following result is true for *rational* β and, in particular, for *real* λ ([95]):

Theorem 5.2 *Suppose that either $|\lambda| > m^2$ or $s > 2\rho$. Then $f(z)$ is unbounded on any ray, and one can find a geometric progression $r_n = q^n r_0$, ($r_0 > 0$), for which the limit (52) exists and $L > 0$.*

The above results are based on the following theorem on asymptotics of $f(z)$ along spirals (geometric progressions) of the form $z_n = \lambda^n z_0$, where $z_0 = r_0 e^{i\theta_0} \in \mathbb{C}$ will be specified below.

Consider the Poincaré equation

$$f(\lambda z) = P(f(z)), \quad (53)$$

where

$$P(z) = z^m + p_{m-1}z^{m-1} + \dots + p_1z + p_0. \quad (54)$$

Denote

$$K = \max\{|p_0|, \dots, |p_{m-1}|\}. \quad (55)$$

Suppose that $z_0 = r_0 e^{i\theta_0} \in \mathbb{C}$ is such a point, that

$$|f(z_0)| > \max\{e, 2mK\}. \quad (56)$$

Let us use the following notations

$$z_n = \lambda^n z_0; \quad \varphi(z) = \log |f(z)|; \quad \rho = \frac{\log m}{\log |\lambda|} \quad (57)$$

in part already introduced earlier. (It follows from (56) that $\varphi(z_0) > 1$)

Theorem 5.3 *Suppose that (56) is satisfied. Then the limit along the spiral $z_n = \lambda^n z_0$*

$$\lim_{n \rightarrow \infty} \frac{\varphi(z_n)}{|z_n|^\rho} = L(z_0) \quad (58)$$

exists and

$$\frac{\varphi(z_0) - 3Km/(2|f(z_0)|)}{|z_0|^\rho} < L(z_0) < \frac{\varphi(z_0) + 3Km/(2|f(z_0)|)}{|z_0|^\rho}. \quad (59)$$

In particular, in view of (56)

$$\frac{\varphi(z_0) - 3/4}{|z_0|^\rho} < L < \frac{\varphi(z_0) + 3/4}{|z_0|^\rho}. \quad (60)$$

Furthermore, L is a continuous function on the domain $\{z \in \mathbb{C} \mid |f(z)| > \max(e, 2mK)\}$.

Remark 5.1 *Here we deal mainly with the asymptotics of entire solutions of (50) However some statements are valid for arbitrary solutions of (50). In particular, for validity (58) and (59) no assumptions on the smoothness are needed.*

5.6. Asymptotics and dynamics in the real case. Properties of the Julia set

Further refinements are possible when $\lambda > 1$ is real and $P(z) = p_m z^m + \dots + p_1 z + p_0$ is a polynomial with real coefficients.

Without loss of generality we can assume also that:

$$f(0) = P(0) = 0; \quad P'(0) = \lambda > 1 \text{ and } f'(0) = 1$$

For reader convenience, we recall now some basic notions from the iteration theory and complex dynamics [25, 110].

To make things shorter, we give here definitions which slightly differ from standard ones, but are equivalent to them in the *polynomial* case. That will suffice our needs, in what follows.

We will especially need the component of ∞ of Fatou set $\mathcal{F}(P)$ given by

$$\mathcal{F}_\infty(P) = \left\{ z \in \mathbb{C} \mid \lim_{n \rightarrow \infty} P^{(n)}(z) = \infty \right\}, \quad (61)$$

The filled Julia set is given by

$$\mathcal{K}(p) = \left\{ z \in \mathbb{C} \mid (P^{(n)}(z))_{n \in \mathbb{N}} \text{ is bounded} \right\} = \mathbb{C} \setminus \mathcal{F}_\infty(P). \quad (62)$$

Now, according to [110] one can define the Julia set $\mathcal{J}(p)$, as

$$\partial \mathcal{K}(P) = \partial \mathcal{F}_\infty(P) = \mathcal{J}(P). \quad (63)$$

In the case of polynomials this can be used as an equivalent definition of the Julia set.

Let $f(z)$ be an entire solution of (50). In contrast with the previous section (where asymptotics of $|f(z)|$ is studied) here we collect some results on the asymptotics of the solution $f(z)$ *itself* (in some angular regions of the complex plane). Our presentation is based on ([69, 87, 95]).

Theorem 5.4 ([69, Theorem 1]) *Let f be an entire solution of the functional equation (50). Furthermore, suppose that \mathcal{F}_∞ , the Fatou component of ∞ of P , contains an angular region of the form*

$$W_\beta = \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \beta\}$$

for some $\beta > 0$. Then for any $\varepsilon > 0$ and any $M > 0$ the asymptotic relation

$$f(z) = \exp \left(z^\rho Q \left(\frac{\log z}{\log \lambda} \right) + o(|z|^{-M}) \right) \quad (64)$$

holds uniformly for $z \in W_{\beta-\varepsilon}$, where Q is a periodic holomorphic function of period 1 on the strip $\{w \in \mathbb{C} \mid |\Im w| < \frac{\beta}{\log \lambda}\}$. The real part of $z^\rho Q(\frac{\log z}{\log \lambda})$ is bounded between two positive constants; Q takes real values on the real axis.

Remark 5.2 *Notice that the condition on the Fatou component \mathcal{F}_∞ is used in the proof of this theorem to ensure that $f(z)$ tends to infinity in the angular region W_β . Therefore, this condition could be replaced by*

$$\lim_{z \rightarrow \infty} f(z) = \infty \text{ for } |\arg z| < \beta.$$

Yet a stronger result can be derived under the additional assumption that the Julia set $\mathcal{J}(P)$ of polynomial $P(z)$ is *real* (see Corollary 5.1, below).

Remark 5.3 *The latter assumption on the reality of $\mathcal{J}(P)$ looks artificial at the first glance, but, in fact is typical in applications, related to the diffusion on fractals. The rough explanation of the latter fact is the following:*

The zeros of the solution $f(z)$ for (50) are eigenvalues of the infinitesimal generator of the diffusion, i.e. the “Laplacian on the fractal”. The latter operator is self-adjoint, and its eigenvalues are real. Therefore zeros of $f(z)$ are real and, finally, this implies that Julia set of polynomial $P(z)$ should be real. This motivates our special interest for Poincaré equations with polynomial $P(z)$, having real Julia set.

Corollary 5.1 ([87, Corollary 4.1]) *Assume that P is a real polynomial such that $\mathcal{J}(P)$ is real and all coefficients p_i ($i \geq 2$) of P are non-negative. Then $\mathcal{J}(P) \subset \mathbb{R}^- \cup \{0\}$ and therefore*

$$f(z) \sim \exp \left(z^\rho Q \left(\frac{\log z}{\log \lambda} \right) \right) \quad (65)$$

for $z \rightarrow \infty$ and $|\arg z| < \pi$. Here Q is a periodic function of period 1 holomorphic in the strip given by $|\Im w| < \frac{\pi}{\log \lambda}$. Furthermore, for every $\varepsilon > 0$ the real part of $z^\rho Q(\frac{\log z}{\log \lambda})$ is bounded between two positive constants for $|\arg z| \leq \pi - \varepsilon$.

If $P(z)$ is a quadratic polynomial (a case, arising in numerous applications) it is possible to give an exact criterion for reality of $\mathcal{J}(P)$:

Lemma 5.1 ([95, Lemma 6.7]) *Let*

$$P(z) = az(z - \omega), \quad 0 \neq \omega \in \mathbf{R} \quad (66)$$

Then Julia set $\mathcal{J}(P)$ is real, if and only if the following condition is fulfilled

$$a|\omega| \geq \begin{cases} 2, & \omega > 0 \\ 4, & \omega < 0 \end{cases} \quad (67)$$

It would be interesting to find some constructive *conditions for reality* of $\mathcal{J}(P)$ in terms of coefficients of P (at least for polynomial of small order in the beginning).

Note also, that in the above Lemma $|P'(0)| = a|\omega|$ and $m = \deg P(z) = 2$.

Therefore (67) can be rewritten in the form:

$$|P'(0)| \geq \begin{cases} m, & \omega > 0 \\ m^2, & \omega < 0 \end{cases} \quad (68)$$

It turns out that the *necessity* part of latter criterion (in the form (68)) is valid for general polynomial P , with *real* Julia set $\mathcal{J}(P)$. Namely, the following result of Pommerenke–Levin–Eremenko–Yoccoz type (on inequalities for multipliers) is true:

Theorem 5.5 ([87, Theorem 4.1]) *Let P be a polynomial of degree $m > 1$ with real Julia set $\mathcal{J}(P)$. Then for any fixed point ξ of P with $\min \mathcal{J}(P) < \xi < \max \mathcal{J}(P)$ we have $|P'(\xi)| \geq m$. Furthermore, $|P'(\min \mathcal{J}(P))| \geq m^2$ and $|P'(\max \mathcal{J}(P))| \geq m^2$. Equality in one of these inequalities implies that P is linearly conjugate to the Chebyshev polynomial T_m of degree m .*

Remark 5.4 *This theorem can be compared to [105, 108, 111, 112] where inequalities (of the opposite direction) for the multipliers of P with connected Julia sets were derived.*

In the foregoing we dealt mainly with the asymptotics of solutions for the polynomial Poincaré equation, only.

The study of the asymptotic behaviour of meromorphic solutions of the general Poincaré equation

$$f(\lambda z) = R(f(z)), \quad z \in \mathbf{C},$$

where $R(z)$ is rational function (rather than polynomial $P(z)$) still, to large extent, remains an open challenge.

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